Problem 2E,7
Suppose F is a closed bounded subset of $\mathbb{R}$ and $g_1, g_2, \ldots$ is an increasing sequence of continuoous real-valued functions on F such that $\sup\{g_1(x), g_2(x), \ldots\} < \infty$ for each $x \in F$. Define a real-valued function g on F by

$$g(x) = \lim_{k\to\infty} g_k(x).$$

Prove that g is continuous on F if and only if $g_1, g_2, \ldots$ converges uniformly on F to g.

Proof: Obviously, if the convergence is uniform, g is continuous. Conversely, asumme that g is continuous. For any $x \in F$, there exists an positive integer N_x such that $|g(x) - g_{N_x}(x)| < \epsilon$. Since these functions are continuous, there exists some $\delta_x > 0$ such $|g(y) - g_{N_x}(y)| < \epsilon$ holds for any $y \in (x - \delta_x, x + \delta_x)$ holds. By F is closed and bounded, we can take a finite cover $B_{x_i}(\delta_{x_i}), i = 1, \ldots, k$ of F. Take $N = \max N_1, \ldots, N_k$, then for $n > N$, we have for any $x \in F$, $|g_n(x) - g(x)| < \epsilon$. This proves the uniform convergence by definition.

Problem 2E,8
Suppose μ is the measure on $(\mathbb{Z}^+, 2^{\mathbb{Z}^+})$ defined by

$$\mu(E) = \sum_{n\in E} \frac{1}{2^n}$$

Prove that for every $\epsilon > 0$, there exists a set $E \subset \mathbb{Z}^+$ with $\mu(\mathbb{Z}^+ - E) < \epsilon$ such that $f_1, f_2, \ldots$ converges uniformly on E for every sequence of functions $f_1, f_2, \ldots$ from $\mathbb{Z}^+$ to $\mathbb{R}$ that converges pointwise on $\mathbb{Z}^+$.

Proof: Choose N large enough such that $\frac{1}{2^N} < \epsilon$. Set $E = \{1, 2, \ldots, N\}$,depening only on ϵ, then $\mu(\mathbb{Z}^+ - E) < \epsilon$. Then since E is finite set, so we have that $f_1, f_2, \ldots$ converges uniformly on E for every sequence of functions $f_1, f_2, \ldots$ from $\mathbb{Z}^+$ to $\mathbb{R}$ that converges pointwise on $\mathbb{Z}^+$.

Problem 3A,3
Suppose $(X, \mathcal{S}, \mu)$ is a measurable space and $f : X \to [0, \infty]$ is an $\mathcal{S}$-measurable function. Prove that

$$\int f d\mu > 0 \; if \; and \; only \; if \mu(\{x \in X : f(x) > 0\}) > 0$$

Proof. Note that $\{x \in X : f(x) > 0\} = \lim_{k\to\infty}\{x \in X : f(x) > \frac{1}{k}\}$. Thus

$$\mu\{x \in X : f(x) > 0\} = \lim_{k\to\infty} \mu\{x \in X : f(x) > \frac{1}{k}\}.$$

If $\int f d\mu = \int_{f>0} f d\mu > 0$, then obvioully $\mu(\{x \in X : f(x) > 0\}) > 0$ for otherwise we are doing integration on a null set.

If $\mu(\{x \in X : f(x) > 0\}) > 0$, then there exists k_0 such that $\mu\{x \in X : f(x) > \frac{1}{k_0}\} > 0$, so

$$\int f d\mu \geq \int_{f>\frac{1}{k_0}} f d\mu \geq \frac{1}{k_0}\mu\{x \in X : f(x) > \frac{1}{k_0}\} > 0.$$

□

Problem 3A,4
Give an example of a Borel measurable function $f : [0, 1] \to (0, \infty)$ such that $L(f, [0, 1]) = 0$.

Proof: Define $f(x) = 1$ if x is not a rational number; $f(x) = \frac{1}{n}$ if $x = \frac{m}{n}, (m, n) = 1$. In order to check $L(f, [0, 1]) = 0$, we only need to check that for any interval $(a, b) \subset [0, 1]$, the infimum of f on it is 0. For any $\epsilon > 0$, there exists n with $\frac{1}{n} < \epsilon$. Note that the rational number of the form $\frac{m}{l}, (m, l) = 1, l \leq n$ is finite. But there are infinite rational number on the interval (a, b).This completes the proof.

Problem 3A,8
Suppose λ denotes the Lebesgue measure on $\mathbb{R}$. give an example of a sequence $f_1, f_2, \ldots$ of simple Borel measurable function from $\mathbb{R}$ to $[0, \infty)$ such that $\lim_{k\to\infty} f_k(x) = 0$ for every $x \in \mathbb{R}$ but $\lim_{k\to\infty} \int f_k d\lambda = 1$.

Proof: Define $A = \int_{\mathbb{R}} \exp^{-x^2} dx$. Set $f_k(x) = \frac{1}{kA} \exp^{-\frac{x^2}{k^2}}$.

Problem 3B,4

- Supose $(X, \mathcal{S}, \mu)$ is a measurable space with $\mu(X) < \infty$. Suppose that f is a bounded measurable function. Prove

$$\int f d\mu = \inf\{\sum_{j=1}^{m} \mu(A_j) \sup_{A_j} f : A_1, \ldots, A_m \text{ } is\ an\ \mathcal{S} - partition\ of\ X\}$$

- Show that the conclusion of part a can fail if the hypothesis that f is bounded is replaced by $\int f < \infty$.
- Show that the conclusion of part a can fail if the condition $\mu(X) < \infty$ is deleted.

Proof.

- Obviously

$$\int f d\mu \leq \inf\{\sum_{j=1}^{m} \mu(A_j) \sup_{A_j} f : A1, \ldots, A_m \text{ } is\ an\ \mathcal{S} - partition\ of\ X\}$$

  On the other hand, we can find a sequence of simple function f_k satisfying $f_k(x) \geq f(x)$ and f_k converges uniformly to f. Note that

$$\int f_k d\mu \geq \inf\{\sum_{j=1}^{m} \mu(A_j) \sup_{A_j} f : A1, \ldots, A_m \text{ } is\ an\ \mathcal{S} - partition\ of\ X\}$$

  and by dominated convergence theorem,

$$\int f d\mu \to \int f_k d\mu.$$

  Thus

$$\int f d\mu \geq \inf\{\sum_{j=1}^{m} \mu(A_j) \sup_{A_j} f : A1, \ldots, A_m \text{ } is\ an\ \mathcal{S} - partition\ of\ X\}.$$

- Take $((0,1), \mathcal{L}, \lambda)$ be our standard Lebesgue measure space and $f(x) = \frac{1}{\sqrt{x}}$. Then $\int f d\lambda < \infty$. But for any $\mathcal{L}$-partition $A_1, \ldots, A_m$, there always exists some A_i such that $\sup_{A_j} f = \infty$. Thus

$$\inf\{\sum_{j=1}^{m} \mu(A_j) \sup_{A_j} f : A_1, \ldots, A_m \text{ } is\ an\ \mathcal{S} - partition\ of\ X\} = \infty.$$

- Take $((1, \infty), \mathcal{L}, \lambda)$ be our standard Lebesgue measure space and $f(x) = \frac{1}{x^2}$. Then for any $\mathcal{L}$-partition $A_1, \ldots, A_m$, there always exists some A_i such that $\lambda(A_i) = \infty$. Thus,

$$\inf\{\sum_{j=1}^{m} \mu(A_j) \sup_{A_j} f : A_1, \ldots, A_m \text{ } is\ an\ \mathcal{S} - partition\ of\ X\} = \infty.$$

□

Problem 3B,5
Let λ be the Lebesgue measura on $\mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function such $\int |f| d\lambda < \infty$. Prove that

$$\lim_{k\to\infty} \int_{[-k,k]} f d\lambda = \int f d\lambda$$

Proof: Apply the dominated convergence theorem to $f\chi_{[-k,k]}$. Note $|f|$ is the dominated function.

Problem 3B,7
Let λ be the Lebesgue measura on $\mathbb{R}$. Give an example of a continuous function $f : (0,1) \to \mathbb{R}$ such that $\lim_{n\to\infty} \int_{(\frac{1}{n},1)} f d\lambda$ exists but $\int_{(0,1} f d\lambda$ is not defined.

Proof: By a change of variable and translation, we consider to construct such example on $(0,\infty)$ such that $\int_{(0,\infty)} f d\lambda$ does not exist but $\lim_{k\to\infty} \int_{(0,2k)} f d\lambda$ exists and equal to 0. Just define that for each positive integer k, $f(x) = 1$ for $x \in (2k-2, 2k-1]$;$f(x) = -1$ for $x \in (2k-1, 2k]$.